Eur. Phys. J. A **5**, 263–273 (1999)

THE EUROPEAN PHYSICAL JOURNAL A

Effect of thermal ground state correlations on the statistical properties of the Lipkin model

A.I. Vdovin, A.N. Storozhenko

Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Russia

Received: 4 January 1999 / Revised version: 10 March 1999
Communicated by P. Schuck

Abstract. The renormalized random phase approximation for hot finite Fermi systems is evaluated with the use of the thermo field dynamics formalism. This approximation treats vibrations of a hot finite Fermi system as harmonic ones but takes into account the Pauli principle in a more proper way than the usual thermal RPA, thus incorporating a new type of correlations in a thermal ground state. To demonstrate advantages of the approximation and to analyze a range of its validity, it is applied to the exactly solvable Lipkin model. A comparison is made with the exact grand canonical ensemble calculations, results of the thermal Hartree – Fock approximation and the thermal random phase approximation. The intrinsic energy of the system, the heat capacity, the average value of the quasispin operator z-projection and the particle number variance are calculated as functions of temperature. On the whole, the thermal renormalized RPA appears to be a better approximation than the other two. Its advantage is especially evident in the vicinity of the phase transition point. It is found that within TRRPA the phase transition occurs at lower temperature than in THFA and TRPA.

PACS. 21.60.Jz Hartree-Fock and random-phase approximations – 11.10.Wx Finite-temperature field theory

1 Introduction

The possibility for giant collective vibrations to be built on nuclear excited states was anticipated in 1955 by D.M. Brink in his thesis work [1]. It seems natural that a compound nuclear state could also serve as a "head" state for a giant resonance. This hypothesis was confirmed in 1981 by experiments where a giant dipole resonance was discovered in a hot nucleus produced in heavy ion collisions [2]. Axel-Brink's hypothesis implies that the structure of a "head" state doesn't affect the properties of a giant resonance built on it. It appeared to be true only in some cases. In a particular case of a giant resonance in a hot nucleus some of the GR characteristics (e.g. a width) are dependent on temperature of a compound state. From the theoretical point of view it means that to explain a collective properties of a hot nucleus, it is quite important to elaborate adequate theoretical methods of describing a compound state.

A standard technique of treating quantum many-body systems at finite temperature T is the temperature-dependent Green function method. But in the early seventies a different technique called "thermo field dynamics" (TFD) was introduced [3]. The TFD approach has two obviously appealing features: a) temperature effects arise explicitly as T-dependent vertices, providing a good starting point for various approximations; b) generalization to the time-dependent situation is easy, since temperature and time are independent variables in TFD. Both the features allow for straightforward extensions of well-established zero-temperature approximations, as it was already demonstrated in [4-7].

The main idea behind TFD is a construction of a field theory in which the grand canonical statistical average of a quantity A is given by some sort of expectation value rather than the trace operation

$$\begin{aligned} \ll A \gg &= \frac{1}{Tr(\exp(-H/T))} Tr\left[A \ \exp(-H/T)\right] \\ &= \langle 0(T)|A|0(T)\rangle \ . \end{aligned}$$

TFD gives rigorous prescriptions how to construct a representation in which the "vacuum" expectation value coincides with the statistical average. This aim is achieved by a formal doubling of the Hilbert space of a system. One introduces a fictitious system which is of exactly the same structure as the physical one under consideration. The whole Hilbert space of a hot system is spanned by the direct product of the eigenstates of the Hamiltonian $H|n\rangle = E_n|n\rangle$ and those of the "tilde" Hamiltonian having the same eigenvalues $\widetilde{H}|\tilde{n}\rangle = E_n|\tilde{n}\rangle$. With the doubling of the Hilbert space one can write the expression for a "vacuum" state $|\Psi_0(T)\rangle$ that is called "the thermal vacuum state"

$$|\Psi_0(T)\rangle = \frac{1}{\sqrt{Tr(\exp(-H/T))}} \times \sum_n \exp(-\frac{E_n}{2T})|n\rangle \otimes |\tilde{n}\rangle \,. \quad (1)$$

The vectors $|n\rangle$ and $|\tilde{n}\rangle$ appear as a pair and the function of $|\tilde{n}\rangle$ is merely to pick up the diagonal element of A. A tilde conjugate operator $\widetilde{A}$ acting in the tilde space is associated with any operator A acting in ordinary space in accordance with special rules (see, e.g., [3,4]). The time - translation operator in the hot system appears to be a thermal Hamiltonian $\mathcal{H}$ defined as $\mathcal{H} = H - \widetilde{H}$. Properties of the system excitations are obtained by the diagonalization of $\mathcal{H}$. The thermal vacuum is an eigenstate of $\mathcal{H}$ with a zero eigenvalue. Thus, the dynamical development of the system is carried by the thermal Hamiltonian while the thermal behaviour is controlled by the thermal vacuum.

The TFD approach provides transparent interpretation of collective motion in a hot Fermi - system. The thermal vacuum can be regarded as a temperature - dependent wave function of a compound nuclear state. A collective excitation in a hot nucleus is produced by applying the corresponding collective operator (phonon operator) to this thermal vacuum state. Properties of the collective excitation are dependent on the vacuum correlations.

There are two well known approximations of treating a hot finite Fermi - system: the thermal Hartree - Fock approximation (THFA) and the thermal random phase approximation (TRPA). The latter was applied to study a giant dipole resonance in hot nuclei in a number of papers [8] (see also the review paper [9] and references therein). Also some approximations going beyond TRPA were considered but merely a coupling of thermal particle-hole or TRPA phonon excitations with more complex ones was studied [5,6,10,11].

In the paper [7], a thermal approximation of another type was suggested which is still based on collective harmonic-vibration scheme but treats more accurately thermal vacuum state correlations. The approximation was named the thermal renormalized RPA (TRRPA). Actually, in [7] an idea by Ken ji-Hara [12] has been explored. This kind of approaches has been known for a long time for cold nuclei [13,14] and has recently been applied to calculate various nuclear properties [15]. Later on, a more general and consistent formulation of the approximation for a hot system was given [11,16]. But up to now a range of validity of TRRPA has not been analyzed and its advantages have not been clearly demonstrated. In this paper, we apply TRRPA to the exactly solvable Lipkin model [17] and compare the approximation with the results of exact calculations as well as with THFA and TRPA. A part of the present results has been published in [18].

The organization of the paper is as follows. A general formulation of TRRPA combined with some elements of the TFD formalism are given in Sect. 2. In Sect. 3 we specify the TRRPA formulae for the Lipkin model. The results of the three approximate methods (TRRPA, TRPA and THFA) and a comparison with the exact ones are discussed in Sect. 4. Section 5 contains the summary and concluding remarks.

2 Thermal renormalized RPA. General formulation

Let us consider a system of N-fermions coupled through two-body interactions. The Hamiltonian is written as

$$H = \sum_{12} t_{12} a_1^+ a_2 + \frac{1}{4} \sum_{1234} V_{1234} a_1^+ a_2^+ a_4 a_3 \,, \quad (2)$$

where a^+ and a are fermion creation and annihilation operators. The one-body part

$$t_{12} = T_{12} - \lambda \delta_{12}$$

contains the kinetic energy matrix T_{12} as well as the chemical potential λ.

To describe thermal properties of the system within the TFD formalism, one needs to construct the thermal Hamiltonian $\mathcal{H} = H - \widetilde{H}$ and then find the corresponding thermal vacuum state $|\Psi_0(T)\rangle$ as an eigenstate of $\mathcal{H}$ with a zero eigenvalue. To solve this problem in a full scale is obviously impossible. Due to this one has to introduce some approximations. A good starting point is the thermal mean field approximation.

Usually, under statistical consideration of an excited finite Fermi system, the unified statistically averaged mean field potential is used for all energy states (instead of finding the best single - particle approximation for each of the states appearing in a partition function). A single - particle Hamiltonian H_0 corresponding to this potential can be written as

$$H_0 = U_0 + \sum_1 \varepsilon_1 \alpha_1^+ \alpha_1 \,,$$

where U_0 is a constant and ε_1 - energies of single - particle states. According to Bogoliubov's variational theorem [19], one can define for a general Hamiltonian (2) a model grand thermodynamic potential

$$\Omega_{mod}(H) = \Omega_0 + \ll H - H_0 \gg_0 \quad (3)$$

which is an upper limit of the "true" grand potential $\Omega(H)$: $\Omega_{mod}(H) \geq \Omega(H)$. Here

$$\Omega_0 = \ll H_0 \gg_0 - TS_0 \,,$$

$$\ll H_0 \gg_0 = \frac{Tr\left[H_0 \exp\left(-\frac{H_0}{T}\right)\right]}{Tr\left[\exp\left(-\frac{H_0}{T}\right)\right]}$$

The density matrix of the system with the Hamiltonian H_0 is

$$n_1 = \ll \alpha_1 \alpha_1^+ \gg_0 \,,$$

and the following expression for the entropy S_0 is valid

$$S_0 = -\sum_1 \left[n_1 \ln(n_1) + (1 - n_1)\ln(1 - n_1)\right] \,.$$

The model thermodynamic potential Ω_0 can be constructed with any single - particle Hamiltonian H_0. But, of course, a choice of the best possible single - particle approximation which would correspond to the minimal difference $\Omega_{mod}(H) - \Omega(H)$ is of special interest. The single-particle Hamiltonian which provides such a minimization is called a self-consistent Hartree-Fock Hamiltonian. To find the corresponding single-particle basis one should make a unitary transformation D ($DD^+ = I$) of the Hamiltonian (2) from the initial particle operators a_1^+, a_1 to new HF quasiparticle operators α_1^+, α_1

$$a_1^+ = \sum_2 D_{21}^* \alpha_2^+ \,, \qquad a_1 = \sum_2 D_{21} \alpha_2 \,. \tag{4}$$

Unitarity of the transformation provides a conservation of the commutation rules. Since $\Omega_{mod}(H)$ is an upper limit for $\Omega(H)$ equations for the coefficients D can be derived from the requirement that $\Omega_{mod}(H)$ is minimal under the constraint $DD^+ = I$. These equations are the following:

$$\begin{aligned}
&\frac{\delta}{\delta D_{12}^*}\left(\Omega_{mod} - \sum_2 \xi_2 \sum_3 D_{23}^* D_{23}\right) \\
&= \frac{\delta}{\delta D_{12}^*}\left(\ll H \gg_0 - \sum_2 \xi_2 \sum_3 D_{23}^* D_{23}\right) = 0 \\
&\frac{\delta}{\delta D_{12}}\left(\Omega_{mod} - \sum_2 \xi_2 \sum_3 D_{23}^* D_{23}\right) \\
&= \frac{\delta}{\delta D_{12}}\left(\ll H \gg_0 - \sum_2 \xi_2 \sum_3 D_{23}^* D_{23}\right) = 0 \,,
\end{aligned} \tag{5}$$

which should be completed by the number conserving condition

$$\sum_1 \ll a_1^+ a_1 \gg_0 = N \,.$$

Then single-particle energies ε_1 are obtained from the equations

$$\begin{aligned}
\frac{\delta \Omega_{mod}}{\delta n_1} &= \frac{\delta}{\delta n_1}\Big(\ll H - H_0 \gg_0 \\
&\quad + T \sum_1 \left[n_1 \ln(n_1) + (1-n_1)\ln(1-n_1)\right]\Big) \\
&= 0
\end{aligned} \tag{6}$$

or

$$\varepsilon_1 = \frac{\delta \ll H \gg_0}{\delta n_1} \,,$$

if a function n_1 has the Fermi - Dirac form

$$n_1 = \left[1 + \exp\left(\frac{\varepsilon_1}{T}\right)\right]^{-1} \,. \tag{7}$$

Since in TFD the statistical average is equal to the expectation value of the corresponding operator with respect to the thermal vacuum state, the variational procedure should be be reformulated. Namely, let us find a single-particle basis that diagonalizes the single - particle part of the Hamiltonian (2) normally ordered with respect to the thermal vacuum state $|\Psi_0\rangle$. Using Wick's theorem one can rewrite (2) in the form

$$H = H_{00} + H_{11} + H_{22}$$

$$\begin{aligned}
H_{00} &= \sum_{12} t_{12}\rho_{21} + \frac{1}{2}\sum_{1234} V_{1234}\rho_{31}\rho_{42} \,, \\
H_{11} &= :\sum_{12} t_{12} a_1^+ a_2 + \sum_{1234} V_{1234}\rho_{42} a_1^+ a_3 : \,, \\
H_{22} &= :\frac{1}{4}\sum_{1234} V_{1234} a_1^+ a_2^+ a_4 a_3 : \,.
\end{aligned} \tag{8}$$

where ρ_{ij} is the density matrix

$$\rho_{ij} = \langle \Psi_0(T) | a_j^+ a_i | \Psi_0(T) \rangle \,.$$

The unitary transformation (4) with the coefficients satisfying (5) diagonalizes the part $H_{00} + H_{11}$ of the Hamiltonian (8)

$$H_{HF} \equiv H_{00} + H_{11} = U_0 + \sum_1 \varepsilon_1 \alpha_1^+ \alpha_1 \equiv H_0 \,.$$

Thus, H_{HF} is just the required effective single-particle Hamiltonian H_0. The (5) can be rewritten as

$$\sum_2 \left(t_{12} + \sum_{34} V_{1324}\rho_{43}\right) D_{25} = \varepsilon_5 D_{15} \,. \tag{9}$$

In the present notations the particle number conservation condition takes the form

$$\begin{aligned}
N &= \sum_1 \rho_{11} = \sum_1 \langle \Psi_0(T) | a_1^+ a_1 | \Psi_0(T) \rangle \\
&= \sum_{1k} D_{1k}^* D_{1k} \mathcal{N}_k \,.
\end{aligned} \tag{10}$$

The quasiparticle occupation numbers $\mathcal{N}_k = \langle \Psi_0(T) | \alpha_k^+ \alpha_l | \Psi_0(T) \rangle \delta_{kl}$ can be determined once an explicit expression of the thermal vacuum is specified (see also [11,16]).

The THF Hamiltonian is $\mathcal{H}_{HF} = H_{HF} - \widetilde{H}_{HF}$. It describes a motion of noninteracting heated quasiparticles in a self-consistent mean field.

Within THFA the thermal vacuum state $|0(T)\rangle$ is the eigenvector of the uncorrelated thermal Hamiltonian $\mathcal{H}_{HF}$

$$\begin{aligned}
\left(H_{HF} - \tilde{H}_{HF}\right)|0(T)\rangle &= \left(H_{11} - \tilde{H}_{11}\right)|0(T)\rangle \\
&= \sum_1 \varepsilon_1 \left(\alpha_1^+ \alpha_1 - \tilde{\alpha}_1^+ \tilde{\alpha}_1\right)|0(T)\rangle = 0
\end{aligned}$$

The expression for $|0(T)\rangle$ can be obtained from (1) if one takes as $|n\rangle$ and $|\tilde{n}\rangle$ one-quasiparticle and tilde one-quasiparticle states with the energies ε_1 from (9). But more fruitful is to use the fact that the solution of the

above equation is the vacuum for the following thermal quasiparticle operators $\beta, \tilde{\beta}$:

$$\begin{aligned} \beta_1 &= x_1\alpha_1 - y_1\widetilde{\alpha}_1^+ \\ \tilde{\beta}_1 &= x_1\widetilde{\alpha}_1 + y_1\alpha_1^+ \end{aligned} \tag{11}$$

If the coefficients x_1, y_1 are related to the thermal occupation numbers of the states $\alpha_1^+|0\rangle_{HF}$ with the energies ε_1 as follows

$$x_1 = \sqrt{1-n_1}\,, \quad y_1 = \sqrt{n_1}\,,$$

where n_1 is defined by (7), then

$$\beta_1|0(T)\rangle = \tilde{\beta}_1|0(T)\rangle = 0\,.$$

The transformation $\{x, y\}$ is a unitary one. It is named the thermal Bogoliubov transformation.

The THF Hamiltonian remains diagonal after transformation (11).

$$\begin{aligned} \mathcal{H}_{HF} &= \sum_1 \varepsilon_1 \left(a_1^+ a_1 - \tilde{a}_1^+ \tilde{a}_1\right) \overset{\{x,y\}}{\longrightarrow} \mathcal{H}_{HF} \\ &= \sum_1 \varepsilon_1 \left(\beta_1^+\beta_1 - \tilde{\beta}_1^+\tilde{\beta}_1\right) \end{aligned}$$

Based on the mean field one is able to construct a temperature-dependent Fock space. The Fock space is generated by acting with thermal quasiparticle creation operators $\beta^+, \tilde{\beta}^+$ on the thermal HF vacuum

$$\beta^+|0(T)\rangle, \tilde{\beta}^+|0(T)\rangle, \beta^+\tilde{\beta}^+|0(T)\rangle, ...$$

After construction of the T-dependent Fock space of our system we express the whole thermal Hamiltonian $\mathcal{H} = H - \widetilde{H}$, where H is given in (8), in terms of thermal quasiparticles [11,16]. We get

$$\mathcal{H} = \mathcal{H}_{11} + \mathcal{H}_{22} + \mathcal{H}_{40} + \mathcal{H}_{04} + \mathcal{H}_{31} + \mathcal{H}_{13} \tag{12}$$

where $\mathcal{H}_{mn}$ consists of terms of the type $(\beta^+)^m(\beta)^n$.

A straightforward way to evaluate new approximations is to apply the equation of motion method [13]. Here we start from the thermal RPA since TRRPA can be easily formulated as its natural extension. As for $T = 0$, one has a Raleigh-Ritz variational principle with the thermal Hamiltonian (12)

$$\begin{aligned} &\langle\Psi_0(T)|\left[\delta Q_\nu, [\mathcal{H}, Q_\nu^+]\right]|\Psi_0(T)\rangle \\ &\quad = \omega_\nu\langle\Psi_0(T)|\left[\delta Q_\nu, Q_\nu^+\right]|\Psi_0(T)\rangle \end{aligned} \tag{13}$$

This exact statistical variational problem cannot be solved in practice and the class of variational functions has to be restricted. To choose an appropriate trial wave function let us note that some part of the thermal Hamiltonian (12) can be expressed as a bilinear form of operators $\beta_1^+\tilde{\beta}_2^+$ and $\beta_1\tilde{\beta}_2$. Namely,

$$\begin{aligned} \mathcal{H}_{TRPA} = :& \sum_1 \varepsilon_1(\beta_1^+\beta_1 - \tilde{\beta}_1^+\tilde{\beta}_1) + \\ & \frac{1}{4}\sum_{1234} U_{1234}\Big((x_1y_3\beta_1^+\tilde{\beta}_3^+ + y_1x_3\tilde{\beta}_1\beta_3) \\ & \times(x_2y_4\beta_2^+\tilde{\beta}_4^+ + y_2x_4\tilde{\beta}_2\beta_4)\Big) + \\ & \frac{1}{4}\sum_{1234} U_{1234}\Big((x_1y_3\tilde{\beta}_1^+\beta_3^+ + y_1x_3\beta_1\tilde{\beta}_3) \\ & \times(x_2y_4\tilde{\beta}_2^+\beta_4^+ + y_2x_4\beta_2\tilde{\beta}_4)\Big) :, \end{aligned} \tag{14}$$

where

$$U_{1234} = \sum_{5678} V_{5678} D_{51}^* D_{62}^* D_{73} D_{84}\,.$$

The expression for $\mathcal{H}_{TRPA}$ prompts the following form of a trial wave function (phonon wave function)[1]:

$$\begin{aligned} Q_\nu^+ &= \sum_{12} \psi_{12}^\nu \beta_1^+\tilde{\beta}_2^+ - \phi_{12}^\nu\tilde{\beta}_2\beta_1 \\ &= \sum_{12} \psi_{12}^\nu A_{12}^+ - \phi_{12}^\nu A_{12} \end{aligned} \tag{15}$$

If, in addition, the exact thermal vacuum in (13) is replaced by the THF vacuum state $|0(T)\rangle$, the exact matrix equation (13) reduces to the TRPA equations for the amplitudes $\psi_{24}^\nu, \phi_{24}^\nu$ and the excitation energy ω_ν

$$\begin{aligned} &\varepsilon_{24}\psi_{24}^\nu + \frac{1}{2}x_2y_4\sum_{13} U_{1234}(x_1y_3\phi_{13}^\nu + x_3y_1\psi_{31}^\nu) \\ &\quad - \frac{1}{2}x_4y_2\sum_{13} U_{1234}(x_3y_1\phi_{13}^\nu + x_1y_3\psi_{31}^\nu) = \omega_\nu\psi_{24}^\nu \end{aligned}$$

$$\begin{aligned} &\varepsilon_{42}\phi_{42}^\nu + \frac{1}{2}x_4y_2\sum_{13} U_{1234}(x_1y_3\phi_{13}^\nu + x_3y_1\psi_{31}^\nu) \\ &\quad - \frac{1}{2}x_2y_4\sum_{13} U_{1234}(x_3y_1\phi_{13}^\nu + x_1y_3\psi_{31}^\nu) = -\omega_\nu\phi_{42}^\nu\,. \end{aligned}$$

At this stage $\mathcal{H}_{TRPA}$ (14) becomes diagonal in the phonon operators. In TRPA the Q^+, Q operators are boson operators (the same holds for the bifermionic operators A_{12}^+ and A_{12}). The index ν runs over the TRPA solutions with positive and negative energies ω_ν. The negative-energy solutions appear naturally and ensure that the set of thermal phonon states is complete within the TRPA approximation. The thermal vacuum state $|\Psi_0(T)\rangle$ is the vacuum for thermal phonons, i.e. $Q_\nu|\Psi_0(T)\rangle = 0$. In terms of thermal quasiparticle operators, the TRPA vacuum is given by

[1] If the two-body interaction V_{1234} contains the pairing term, the terms $\beta_1^+\beta_2^+$, $\tilde{\beta}_1^+\tilde{\beta}_2^+$ and their hermitian conjugates have to be added to the phonon wave function (15) as it has been done in [4,5] and also in [6]. Of course, in such a case the present consideration has to begin with the thermal Hartree - Fock - Bogoliubov approximation.

$$\Psi_0(T) = \frac{1}{\sqrt{N}} \exp \frac{1}{4} \left[\sum_{\nu} \sum_{1234} (\psi^{-1})^{\nu}_{12} \phi^{\nu}_{34} A^{+}_{12} A^{+}_{34} \right] |0(T)\rangle \ ,$$

where $|0(T)\rangle$ is the THF vacuum.

Now we are ready to formulate the thermal renormalized RPA. The main assumption is that the bifermionic operators A^{+}_{12} and A_{12} are no longer bosonic, as in TRPA, but obey modified commutation relations [12]

$$\left[A_{12}, A^{+}_{34}\right] = \delta_{13}\delta_{24} \left(1 - q_1 - q_2\right) \ ,$$

where q_i are c-numbers which are defined as follows:

$$\langle \Psi_0(T)|\beta^{+}_1 \beta_2|\Psi_0(T)\rangle = \langle \Psi_0(T)|\tilde{\beta}^{+}_1 \tilde{\beta}_2|\Psi_0(T)\rangle = \delta_{12} q_1$$

The thermal vacuum state $|\Psi_0(T)\rangle$ has to be also redefined. It will be done below.

The substitution [12,13]

$$b_{12} = \frac{A_{12}}{\sqrt{1 - q_{12}}} \ , \quad b^{+}_{12} = \frac{A^{+}_{12}}{\sqrt{1 - q_{12}}} \ ,$$

with $q_{12} = q_1 + q_2$ leads to bosonic commutation rules for the operators b_{12}, b^{+}_{12}. By analogy with TRPA, we introduce phonon creation operators as

$$Q^{+}_{\nu} = \sum_{12} \psi^{\nu}_{12} b^{+}_{12} - \phi^{\nu}_{12} b_{12}$$

and define the thermal vacuum state as a vacuum for this new phonons. Then

$$|\Psi_0(T)\rangle = \frac{1}{\sqrt{N}} \exp \frac{1}{2} \sum_{\nu} \sum_{1234} C^{\nu}_{1234} b^{+}_{12} b^{+}_{34} |0(T)\rangle$$

where the matrices C^{ν}_{1234} are defined through

$$\sum_{12} \psi^{\nu}_{12} C^{\nu}_{1234} = \phi^{\nu}_{34} \ .$$

It is more convenient to write the new system of equations in terms of new variables, namely

$$\Psi^{\nu}_{24} = \frac{1}{x_2 y_4 + y_2 x_4} (\psi^{\nu}_{24} - \phi^{\nu}_{42}) \ ,$$
$$\Phi^{\nu}_{24} = \frac{1}{x_2 y_4 - y_2 x_4} (\psi^{\nu}_{24} + \phi^{\nu}_{42}) \ .$$

The result is

$$\varepsilon_{24} \Psi^{\nu}_{24} + \frac{1}{2} \sum_{13} \sum_{5678} (V_{5678} D^{*}_{51} D^{*}_{62} D_{73} D_{84}) \times \sqrt{1 - q_{13}} \sqrt{1 - q_{24}} (n_1 - n_3) \Psi^{\nu}_{31} = \omega_{\nu} \Psi^{\nu}_{24} \quad (16)$$

$$q_1 = \sum_{\substack{\nu > 0 \\ 2}} \left[(\Psi^{\nu}_{12} x_2 y_1)^2 + (\Psi^{\nu}_{21} y_2 x_1)^2 \right] \ , \quad (17)$$

where $\varepsilon_{24} = \varepsilon_2 - \varepsilon_4$. In accordance with its definition the structure of the new thermal vacuum state differs from that of the THF- and TRP- approximations. It allows for a nonvanishing number of thermal quasiparticles, thus incorporating a new kind of particle correlations. In particular, the role of the Pauli principle in its structure is taken into account in a better way than in the TRPA vacuum state. This produces in (16) the blocking factors $\sqrt{1 - q_{ij}}$ that are less than unity and at the first glance effectively reduce the interaction. But since the system of equations (16,17) is nonlinear, a real effect of the new correlations is not so simple.

The (16,17) are coupled to the mean field equations (9) through the one-particle density matrix ρ_{ij} (10). In TRRPA we cut this coupling assuming for the density matrix the same form as in THFA or TRPA (i.e. the values $\mathcal{N}_k$ from (10) are taken to be equal to n_k (7)). The problem is strongly simplified due to this approximation. In particular, the back-dependence of a single-particle motion from the collective amplitudes $\psi^{\nu}_{12}, \phi^{\nu}_{12}$ disappears [2].

3 Application of TRRPA to the Lipkin model

Now we specify the TRRPA equations for the Lipkin model.

This well-known model has been used many times to justify approximate methods of the many-body theory at finite temperature. For example, the works [4,21] have been focused on boson expansion methods and symmetry breaking in a hot Lipkin system. The so-called mixed state representation has been formulated and then applied to the Lipkin model in [21-24]. Within this approach THFA [24] and TRPA [25] have been studied. The Lipkin model has also been used as a testing ground for the static path approximation (SPA) (see, [26]).

We use the version of the Lipkin model with an interaction acting between a pair of particles with parallel spins only. The model system consists of N fermions distributed over two levels with degeneracy Ω ($\Omega = N$). The energy of the lower and upper level is $-\varepsilon/2$ and $+\varepsilon/2$, respectively. Thus, the Hamiltonian has the form

$$H_{LMG} = \varepsilon J_z - \frac{1}{2} V \left(J_+ J_+ + J_- J_-\right) \ , \quad (18)$$

where the operators of quasispin J and its components J_+, J_-, J_z are defined as follows:

$$J^2 = \frac{1}{2} \left(J_+ J_- + J_- J_+\right) + J^2_z \ ,$$

$$J_z = \frac{1}{2} \sum_{p=1}^{\Omega} \left(a^{+}_{2p} a_{2p} - a^{+}_{1p} a_{1p}\right) \ ,$$
$$J_+ = \sum_{p=1}^{\Omega} a^{+}_{2p} a_{1p} \ ,$$
$$J_- = \left(J_+\right)^{+} \ .$$

[2] This back-dependence is the subject of more refined approximations (see, e.g. [16,20])

Indices "1" and "2" label the lower and upper levels, respectively, index p enumerates the sublevels.

At the first step, we do the thermal Hartree – Fock approximation. To this aim, a unitary transformation D (4) is applied to the initial particle operators a^+_{ip}, a_{ip} to transform them into the Hartree - Fock quasiparticle operators $\alpha^+_{ip}, \alpha_{ip}$ and then the thermal Bogoliubov transformation (11) transforms $\alpha, \tilde{\alpha}$ into thermal quasiparticles.

The coefficients of the D and $\{x, y\}$ transformations are determined under the requirement of the minimum of Ω_{mod} (3).

If the coefficients $D_{ii'}$ are parametrized as in [27]

$$D_{11} = D_{22} = \cos\theta, \qquad D_{12} = -D_{21} = \exp(i\varphi)\sin\theta$$

one gets the following expression for the intrinsic energy $E = \ll H \gg_0 = \langle 0(T)|H|0(T)\rangle$:

$$E = \frac{\varepsilon\Omega}{2}(y_2^2 - y_1^2)\left[\cos 2\theta - \frac{\chi_0(y_2^2 - y_1^2)}{2}\sin^2 2\theta\cos 2\varphi\right]$$

where χ_0 is the effective coupling constant

$$\chi_0 = \frac{V(\Omega - 1)}{\varepsilon}\ .$$

Now the entropy of the system is

$$S_0 = -2\Omega\sum_{i=1,2}(y_i^2\ln y_i + x_i^2\ln x_i)\ .$$

After variation of Ω_0 over the variables $\theta, \varphi, x_i, y_i$, taking into account the constraints for x_i, y_i and the particle number conservation condition one finds two different solutions depending on the value of the effective coupling constant χ_0 and temperature T.

The first solution (the normal phase) exists if the T-dependent effective coupling constant $\chi(T) \equiv \chi_0(n_1 - n_2) \leq 1$. This solution corresponds to the following values of variables:

$$\theta = 0,\ \varphi = 0,\ \varepsilon(T) = \varepsilon\,,$$

$$n_{1,2} = y^2_{1,2} = \frac{1}{1 + \exp(\mp\varepsilon/2T)},$$

The thermal ground state energy is

$$E = \frac{\varepsilon\Omega(n_2 - n_1)}{2}$$

and the Hartree - Fock part of the whole thermal Hamiltonian $\mathcal{H}$ does not depend on temperature.

The second solution (the deformed phase) exists when $\chi(T) \equiv \chi_0(n_1 - n_2) > 1$. It corresponds to the values of variables

$$\cos^{-1}2\theta = \chi(T),\ \varphi = 0,\ \varepsilon(T) = \varepsilon\chi(T),$$

and

$$n_{1,2} = y^2_{1,2} = \frac{1}{1 + \exp(\mp\varepsilon(T)/2T)}\ .$$

The energy of the thermal "deformed" ground state is

$$E = \frac{\varepsilon\Omega(n_2 - n_1)}{4}\left(\chi(T) + \frac{1}{\chi(T)}\right)$$

In this regime the thermal Hartree - Fock Hamiltonian appears to be temperature - dependent.

The value of the chemical potential λ is always equal to zero due to the symmetry of the Lipkin system.

After extracting the Hartree – Fock part of the thermal Lipkin Hamiltonian we take into account the interaction of thermal quasiparticles. The following part of $\mathcal{H}_{LMG}$ corresponds to $\mathcal{H}_{TRPA}$ (14)

$$\begin{aligned}\mathcal{H}_{TRPA} = {} & \varepsilon(T)\left(B - \widetilde{B}\right) - \frac{V(n_1 - n_2)(1 + \cos^2 2\theta)}{4} \\ & \times\left[\left(A^+A^+ + AA\right) - \left(\widetilde{A}^+\widetilde{A}^+ + \widetilde{A}\widetilde{A}\right)\right] \\ & + \frac{V(n_1 - n_2)\sin^2 2\theta}{2}\left[A^+A - \widetilde{A}^+\widetilde{A}\right]\ , \quad (19)\end{aligned}$$

where

$$B = \frac{1}{2}\sum_{p=1}^{\Omega}\left(\beta^+_{2p}\beta_{2p} - \beta^+_{1p}\beta_{1p}\right)\ , \qquad A^+ = \sum_{p=1}^{\Omega}\beta^+_{2p}\tilde{\beta}^+_{1p}\,.$$

In accordance with the discussion of Section 2, we assume the following commutation rules for the thermal bi-quasiparticle operators A, A^+, $\tilde{A}$ and $\tilde{A}^+$:

$$[A, A^+] = [\tilde{A}, \tilde{A}^+] = N(1 - q_1 - q_2) \equiv N(1 - 2q)\,. \quad (20)$$

The c-numbers q_i are the numbers of thermal quasiparticles in the temperature - dependent ground state

$$q_i = \frac{1}{N}\langle\Psi_0(T)|N_i^\beta|\Psi_0(T)\rangle = \frac{1}{N}\langle\Psi_0(T)|\tilde{N}_i^\beta|\Psi_0(T)\rangle\,,$$

where $N_i^\beta = \sum_{p=1}^{\Omega}\beta^+_{ip}\beta_{ip}$.

The thermal Hamiltonian (19) is diagonalized in the space of two one-phonon states[3]

$$\begin{aligned}Q_1^+|\Psi_0(T)\rangle &= \left(\psi_1 A^+ - \phi_1 A\right)|\Psi_0(T)\rangle \quad (21)\\ Q_2^+|\Psi_0(T)\rangle &= \left(\psi_2\tilde{A}^+ - \phi_2\tilde{A}\right)|\Psi_0(T)\rangle\,.\end{aligned}$$

The states (21) have to be orthonormalized[4], and taking account of (20) the following constraints on the amplitudes ψ and ϕ are derived:

$$\psi_i^2 - \phi_i^2 = [N(1 - 2q)]^{-1}\ , \quad i = 1, 2\,.$$

The system of equations for ψ_i, ϕ_i and the phonon frequencies ω_i have two solutions. It appears that only a positive value of ω_1 and a negative value of ω_2 are allowed under the requirement that the wave functions $Q_1^+|\Psi_0(T)\rangle$

[3] Due to simplicity of the model there is no sense to introduce now "new bosonic operators" b^+, b, as it was done in Section 2

[4] Note that $Q_1^+ = \widetilde{Q}_2^+$

and $Q_2^+|\Psi_0(T)\rangle$ are vectors of the Hilbert space. The final result is the following:

$$\omega_1 = \omega \equiv \sqrt{\mathcal{E}^2 - \left[\varepsilon\chi(T)\,(1-2q)\,\frac{(1+\cos^2 2\theta)}{2}\right]^2},$$

$$\psi_1^2 = \frac{\mathcal{E}+\omega}{2N\omega(1-2q)}\,, \quad \phi_1^2 = \frac{\mathcal{E}-\omega}{2N\omega(1-2q)}\,,$$

$$\omega_2 = -\omega\,, \quad \psi_2^2 = \psi_1^2\,, \quad \phi_2^2 = \phi_1^2\,,$$

where

$$\mathcal{E} = \varepsilon(T) + \frac{\varepsilon\chi(T)\,(1-2q)\sin^2 2\theta}{2}$$

The equation for q is evaluated with expression (17)

$$q = \frac{1}{2}\frac{\mathcal{E}-\omega}{N\omega} \tag{22}$$

It is interesting to note that in the thermodynamic limit, i.e. as $N \to \infty$, q vanishes and the TRRPA equations are reduced to the TRPA ones.

4 Results and discussion

The numerical calculations are performed for the Lipkin model with N =10 and $\varepsilon = 1$, i.e. we adopt ε as an energy unit.

Firstly, we would like to sketch the exact grand canonical calculations with the Lipkin model. To calculate the grand canonical partition function, one needs the eigenvalues of the model Hamiltonian (18) and degeneracies of the irreducible quasispin representations for different particle numbers from the range $0 < N \leq 2\Omega$. The operators of a quasispin and its projections $J_\pm$ and J_z form the SU(2) algebra, and the quasispin operator commutes with H_{LMG}. So the Hamiltonian matrix breaks up into submatrices Θ_J of dimension $2J+1$. The model Hamiltonian can be diagonalized in each of these subspaces independently. The corresponding eigenvalues are denoted by $E_1^J, E_2^J, \ldots E_{2J+1}^J$. They can easily be calculated analytically (for small N) or numerically (see, e.g. [28]).

To determine the degeneracies of the irreducible quasispin representations Θ_J, we use the results of [24]. A particular distribution of a given number of particles over two degenerate levels can be characterized by the numbers ν_1 and ν_2, where ν_1 is the number of sublevels occupied by particles on both the lower and upper levels; ν_2 is the number of sublevels occupied on neither the lower nor the upper levels. The quasispin J of the state is determined by the distribution of the rest of particles over the $2\tau = \Omega - \nu_1 - \nu_2$ sublevels. The number $2(\tau+\nu_1)$ is equal to the number of particles. The dimension of the subspace of the states with ν_1 occupied and ν_2 empty sublevels is $2^{2\tau}$. There exist $\Omega!/(2\tau)!\nu_i!\nu_2!$ such distinct subspaces for fixed τ and ν_1. Each of them may be decomposed into irreducible subspaces with fixed quasispin values Θ_τ (appearing once), $\Theta_{\tau-1}$ (appearing g_1^τ times) $\Theta_{\tau-2}$ (appearing g_2^τ times), $\ldots$, $\Theta_{\tau-k}$ (appearing g_k^τ times), $\ldots$, $\Theta_{\tau-[\tau]}$ (appearing $g_{[\tau]}^\tau$ times). Here

$$g_k^\tau = \frac{(2\tau)!}{k!(2\tau-k)!} - \frac{(2\tau)!}{(k-1)!(2\tau-k+1)!}.$$

and $[\tau] = \tau$, if τ is integer, $[\tau] = \tau - 1/2$ if τ is half-integer.

Thus, the exact grand partition function of our ensemble is

$$Z(T) = \sum_{\tau\nu_1\nu_2} \frac{\Omega!}{(2\tau)!\nu_1!\nu_2!} \times \sum_k g_k^\tau \sum_m \exp\left[-\frac{E_m^{\tau-k} - 2(\tau+\nu_1)\lambda}{T}\right], \tag{23}$$

More detailed explanations and expressions for grand canonical averages of energy, J_z-projection and the number of particles can be found in [18,24].

We start from the discussion of the T-dependence of ω (Fig. 1). It seems appropriate to distinguish two cases: a) $\chi_0 < 1$; b) $\chi_0 > 1$. A key for understanding a displayed behaviour of $\omega(T)$ is that in the present version of the Lipkin model heating effectively weakens the interaction of particles since at $T \neq 0$ the effective coupling constant χ_0 is multiplied by a thermal factor $n_1 - n_2 < 1$; hence $\chi(T) < \chi_0$ and $\chi(T)$ vanishes when $T \to \infty$. In the case a) the system is in the normal phase at $T = 0$ and stays there when $T \to \infty$. Then $\omega \to \varepsilon$ with increasing T due to vanishing of the effective interaction. A picture is more complicated if $\chi_0 > 1$. Then, the system is in the deformed phase at $T = 0$. In this phase the distance between single-particle levels is proportional to $V(n_1 - n_2)$ and goes down when temperature increases. As a result, the energy ω of the excited state goes down as well. But near the point $T_{cr} = \frac{\varepsilon}{2}\ln^{-1}\frac{\chi_0+1}{\chi_0-1} \simeq 1.0$, where the temperature dependent effective coupling constant $\chi(T) = 1$, the rearrangement of the Hartree – Fock field (i.e. the phase transition) occurs and at $T > T_{cr}$ the system appears to be already in the normal phase. Note that within TRPA the energy ω vanishes at $T = T_{cr}$ whereas within TRRPA

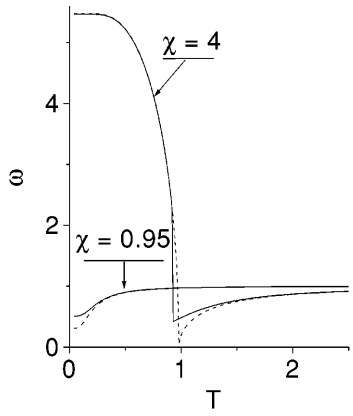

Fig. 1. The energy ω of the lowest excited state as a function of T for two values of the effective coupling constant $\chi_0 = 0.95$ and $\chi_0 = 4.0$. Notation: the TRRPA results – solid lines; the TRPA results – dashed lines

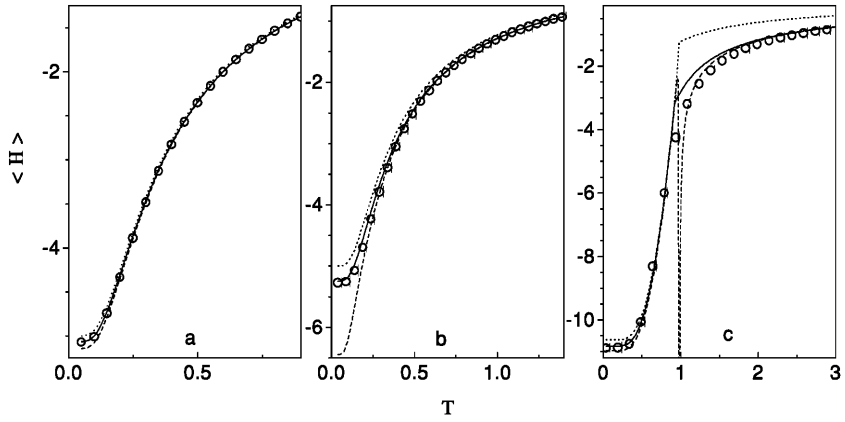

Fig. 2. The average energy $\langle H\rangle$ as a function of temperature T for three values of the effective coupling constant a) $\chi_0 = 0.5$; b) $\chi_0 = 0.95$; c) $\chi_0 = 4.0$. The exact results (the grand canonical ensemble calculations) – open circles; the THF results – short-dashed lines; the TRPA results – long-dashed lines; the TRRPA results – solid lines

ω stays finite. With a further increase in T ω starts to increase and again goes to ε when $T \to \infty$. As one can see in Fig. 1, within TRRPA the phase transition appears at a slightly lower temperature than within TRPA. The reason for this will be discussed later on. A noticeable difference between the TRRPA and TRPA results is just near the critical temperature, and at much lower or higher T both the approximations give close results. Note that a temperature dependence of ω on T within TRPA displayed in Fig. 1 is in complete agreement with that calculated in [21].

Now we discuss T–dependencies of the intrinsic energy $\langle H\rangle$, the quasispin z-projection $\langle J_z\rangle$ and a particle number variance ΔN. The exact values $\langle H\rangle_{GCE}, \langle J_z\rangle_{GCE}$ and ΔN_{GCE} are calculated with the grand canonical partition function (23). The expressions for $\langle H\rangle, \langle J_z\rangle$ and ΔN in TRRPA are obtained by evaluation of the expectation values of the corresponding operators over the thermal vacuum state $|\Psi_0(T)\rangle$. While evaluating $\langle H\rangle_{TRRPA}$ the THF ground state energy E has to be taken into account as well

$$\langle H\rangle_{TRRPA} = \frac{\Omega(n_2-n_1)(1-2q)}{2}\left[\mathcal{E} - \varepsilon\chi(T)\sin^2 2\theta\right] + \frac{(\mathcal{E}-\omega)\,(\varepsilon(T)+\omega)}{2\omega} \times \frac{(n_2-n_1)^2+1}{2(n_2-n_1)}\,. \quad (24)$$

The expression for $\langle H\rangle_{TRPA}$ is followed from (24) if one puts $q = 0$. The THF ground state energy of both the phases has been evaluated in Section 3.

The results for $\langle H\rangle$ and $\langle H\rangle_{GCE}$ are displayed in Fig. 2. Three typical cases are shown: a) a weak coupling case $\chi_0 = 0.5$; b) an intermediate coupling case $\chi_0 = 0.95$; c) a strong coupling case $\chi_0 = 4.0$. At $\chi_0 = 0.5$ the results of TRRPA, TRPA and THFA are very close to each other and to the exact one. Nevertheless the TRRPA curve is closer to the exact result.

The case b) is more interesting (it has already been discussed in [18]). At $\chi_0 = 0.95$ the system is close to the phase transition point. Here, the advantages of TRRPA appear to be most evident. The difference between the approximations is noticeable when $T < 0.3 - 0.5$ and then with the increase in T, results of different approximations approach the exact one. We would like to point out that TRPA overestimates the intrinsic energy of the system. This is because the quasiboson approximation is used together with a variational procedure (see, [27]). Taking into account the Pauli principle more correctly TRRPA removes this drawback of TRPA and improves the description of the ground state energy of a hot system.

In the case c) the system is in the deformed phase at $T = 0$. Although the interaction is strong, the results of TRRPA, TRPA and THFA do not deviate far from each other at $T < T_{cr}$. It means that already THFA is good enough to allow for the main part of thermal quasiparticle correlations and the other approximations give only minor corrections to THFA. But in the vicinity of T_{cr} TRRPA is again the best approximation.

One could already notice that if $\chi_0 > 1$, all the three approximations predict the phase transition in the system with the increase in T. The transition between the two phases manifests itself as a break point of the average energy $\langle H\rangle$ in THFA and TRRPA and as a singular point of this function in TRPA. Thus, within TRPA one deals with a second order phase transition whereas in THFA and TRRPA with a phase transition of the first order. Obviously, the phase transition does not occur in reality, i.e. in exact calculations. This is quite a typical situation when approximate methods are applied to study finite many-body systems. THFA, TRPA and TRRPA predict phase transitions of different characters; also a quality of the description of a system evolution in the vicinity of a critical temperature is different. It is more clearly seen in Fig. 3, where a dependence of a heat capacity C on T is displayed. The heat capacity is calculated as a partial derivative with respect to T of the intrinsic energy (24). At any value of χ_0 the exact heat capacity as a function of T has quite a sharp maximum at $T \sim 0.5$. In the weak interaction case all the three approximations describe $C(T)$ well. But in the cases b) and c) all the approximations demonstrate much sharper behaviour of $C(T)$ in the region of the maximum. Moreover, in the case c) maxima of approximate functions are at noticeably higher temperatures than that of exact one. Note also that $C(T)$ calculated within TRPA has a singular discontinuity at $T = T_{cr}$ whereas the THFA and TRRPA

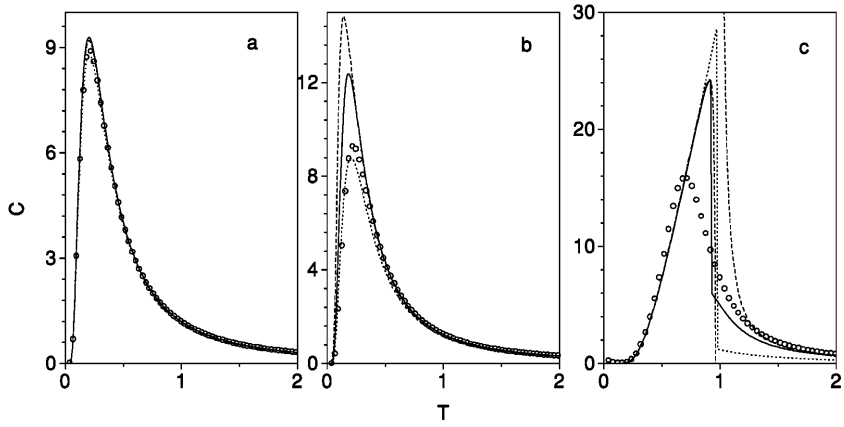

Fig. 3. The heat capacity C as a function of temperature T for three values of the effective coupling constant a) $\chi_0 = 0.5$; b) $\chi_0 = 0.95$; c) $\chi_0 = 4.0$. For notation, see Fig. 2

heat capacities have jump discontinuities (naturally, this is in correspondence with the above-mentioned character of a phase transition in different approximations).

As it is seen in Fig. 2c, the phase transition in TRRPA occurs at slightly lower T than in TRPA and THFA. In the latter approximations the phase transition is at the same value T_{cr} because the rearrangement of the Hartree – Fock field and the collapse of the TRPA collective state are at the same value of $\chi(T)$. Within TRRPA a picture of the phase transition is the following. In the vicinity of T_{cr} one can compare the values of $\langle H\rangle_{TRRPA}$ calculated with the two different mean field configurations corresponding to two phases. It appears that the value of $\langle H\rangle_{TRRPA}$ corresponding to the normal phase remains lower than the value of $\langle H\rangle_{TRRPA}$ calculated with the deformed mean field within a small temperature range at $T < T_{cr}$, where T_{cr} is the critical temperature of the phase transition in THFA [5]. In other words, within TRRPA the normal phase of the system survives in a larger temperature range than within THFA or TRPA. This fact is intimately connected with the behaviour of the collective state energy as a function of the coupling constant. One cannot calculate $\langle H\rangle_{TRPA}$ at $\chi(T) > \chi_{cr}$ because the value of ω is imaginary there. But within TRRPA it is possible because ω remains real and finite at any value of $\chi(T)$. On the whole a curve $C(T)$ calculated within TRRPA is in better agreement with the exact one than the TRPA result. Thus, TRRPA describes the slope of the function $\langle H(T)\rangle$ better than THFA and TRPA.

The expectation value of the operator J_z is proportional to the difference of the numbers of particles on the lower and upper levels of the system. Hence, with the increase in T $J_z \to 0$. But the behaviour of J_z appears to be dependent on χ_0. The expressions for J_z for the different phases have the following forms:

$$\langle J_z\rangle_{TRRPA} = \begin{cases} \dfrac{\Omega(n_2-n_1)(1-2q)}{2} & \text{(Normal phase)} \\ -\dfrac{\varepsilon\Omega(1-2q)}{2V(\Omega-1)} & \text{(Deformed phase)} \end{cases}$$

[5] Let us remember that the normal phase exists on the right-hand side from the phase transition point.

Within THFA and TRPA the expression for $\langle J_z\rangle$ appears to be the same. In the deformed phase $\langle J_z\rangle$ does not depend on temperature. This seems to be the result of the two opposite tendencies. With the increase in T the difference $f_1 - f_2$ (and hence the value $N_1 - N_2$) decreases $n_1 - n_2$ but at the same time the difference of the energies of the single-particle levels decreases (see the corresponding expression for $\varepsilon(T)$ in Sect.3) and this compensates the first effect.

As one can see in Fig. 4, the results of different approximations in the normal phase (the cases a) and b) are very close to each other as well as to the exact one. The largest difference between them is ∼10% at $\chi_0 = 0.95$ and $T \leq 0.1$. Again the results of TRRPA are in better agreement with the exact ones than those of TRPA (and THFA). The worst agreement with the exact result is for the strong coupling case (Fig. 4c). At $T < T_{cr}$ the reason is again a phase transition which is a result of the approximations applied but not the real property of the system. Moreover, in the normal phase, the exact value $\langle J_z\rangle_{GCE}$ goes to zero faster than the approximate one $\langle J_z\rangle_{TRRPA}$. In spite of these discrepancies one can make a conclusion about a qualitative agreement between the exact and approximate results in this case too.

The expression for the particle number variance ΔN_{TRRPA} is

$$\Delta N_{TRRPA} = \sqrt{2Nn_1n_2(1-2q)}\,.$$

This expression is valid in both the normal and deformed phase.

To evaluate this formula, one needs the expectation value with respect to $|\Psi_0(T)\rangle$ of two-body operator $\widehat{N}^2$ where $\widehat{N}$ is the particle number operator in the ordinary space. This matrix element was expanded on the TRRPA phonon (21) basis. Then, in this expansion only the phonon vacuum and two-phonon terms were taken into account. The two-phonon terms give a correction of an order of $\sim \Omega q$. A contribution of four-phonon and even more complex terms seems to be small.

From the expression for ΔN one can see that in TRPA and THFA the particle number fluctuations have only the thermal origin (we deal with the grand canonical ensemble). At the same time, within TRRPA quantum fluctua-

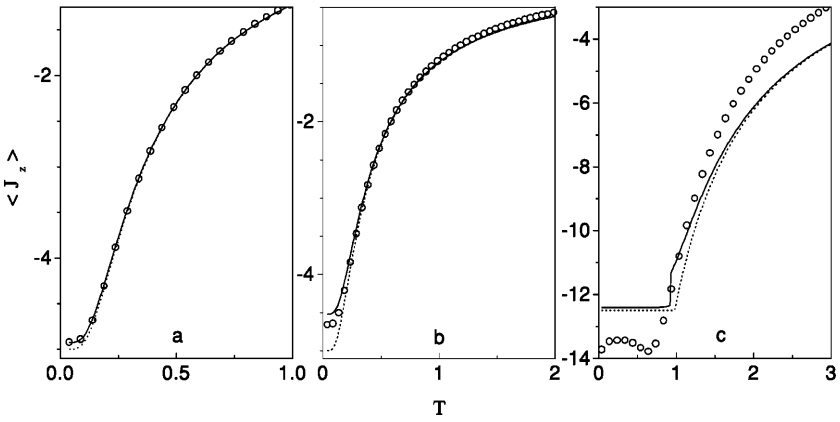

Fig. 4. The average value of the quasispin projection $\langle J_z \rangle$ as a function of temperature T for three values of the effective coupling constant a) $\chi_0 = 0.5$; b) $\chi_0 = 0.95$; c) $\chi_0 = 4.0$ For notation, see Fig. 2

tions exist as well. Note that quantum fluctuations slightly damp thermal ones. The reason for this destructive interference of the two types of fluctuations seems to be the Pauli principle. The nonvanishing q values mean that the single-particle levels are already partially occupied and this is an obstacle for their thermal feeding with the increase in temperature. But the link between the quantum and thermal fluctuations appears to be quite intimate because, when $T \to 0$ the particle variance vanishes, i.e. the quantum fluctuations disappear together with the thermal ones.

In the cases of weak and intermediate couplings the difference between exact and approximate results for ΔN is 2-3%, though the TRRPA results are closer to the exact ones. In the strong coupling case the exact and approximate results noticeably differ only in the vicinity of T_{cr}. Here, TRRPA works also better than the other two approximations (their results coincide with each other).

5 Concluding remarks

The results of the present consideration can be summarized as follows. The thermal renormalized random phase approximation is formulated and for the first time applied to the exactly solvable Lipkin model. The thermodynamic properties of the model are calculated within THFA, TRPA and TRRPA and compared with the exact calculations within the grand canonical ensemble.

On the whole, the results of TRRPA are in better agreement with the exact ones than those of TRPA and THFA. In all cases TRRPA describes quantitatively better the statistical properties of the Lipkin model though in many cases the improvement is not large. Within TRRPA the phase transition appears to be of the first order like in THFA and not of the second order like in TRPA. TRRPA gives a much better description of the intrinsic (or thermal ground state) energy at T slightly above T_{cr}. Actually, the exact calculations do not demonstrate the phase transition and its appearance is a result of the approximations used. But nevertheless TRRPA gives a better description of the system temperature behaviour than the other approximate methods.

The main reason for these TRRPA advantages is the allowance for a nonvanishing number of thermal quasiparticles in the TRRPA thermal vacuum state. Due to this, the role of the Pauli principle is taken into account more properly than in the standard TRPA and a new type of correlations appearing in the thermal vacuum structure affect the statistical properties of the Lipkin model.

The new correlations prevent the collapse of a collective state at large coupling constant, and this appears to be intimately connected with the system behavior in the vicinity of the critical temperature. Though the q value is not large, as a rule it reaches the maximum near and at the phase transition point (see (22) and Fig. 1), and this explains our results. Also, the shift of the phase transition temperature in TRRPA as compared to TRPA (and THFA) seems to be a quite interesting result. This shift pushes the maximum of the heat capacity in the right direction (Fig. 4c).

It is worthwhile to note that with increasing T and N, results of all the approximate methods improve rapidly and at $T \geq 3\varepsilon$ the difference between exact and approximate results is invisible. But it seems that the Lipkin model overestimates decreasing of a two-body interaction with temperature since it does not reproduce appearance of particle-particle and hole-hole excitations at T$\neq$ 0 as it occurs in real nuclear spectra.

Evidently, the present approximation can be improved and some additional corrections can be taken into account. Many suggestions of such an improvement can be found in the literature [13-15], especially in the papers devoted to the self-consistent RPA [16,20]. We suppose to continue our efforts in this direction.

The authors acknowledge an important contribution of Dr. D. Kosov at the early stage of this work and are grateful to him for his interest and correspondence. The comments of Prof. P. Schuck are appreciated. The work was partially supported by the Council of President of Russia on Grants and Support of Leading Scientific Schools and RFBR (grant of RFBR 96-15-96729).

References

1. D.M. Brink, Ph.D. thesis, University of Oxford, 1955
2. J.O. Newton, et al., Phys. Rev. Lett. **46**, 1383 (1981)
3. Y. Takahashi, H. Umezawa, Collective Phenomena **2**, 55 (1975); H. Umezawa, H. Matsumoto, M. Tachiki, *Thermo field dynamics and condensed states*, (North-Holland, Amsterdam - New York - Oxford 1982)
4. T. Hatsuda, Nucl.Phys. A **492**, 187 (1989)
5. K. Tanabe, Phys.Rev. C **37**, 2802 (1988)
6. D.S. Kosov, A.I. Vdovin, Mod. Phys. Lett. A **9**, 1735 (1994)
7. A.V. Avdeenkov, D.S. Kosov, A.I. Vdovin, Mod. Phys. Lett. A **11**, 853 (1996)
8. A.V. Ignatyuk, Izv. AN SSSR, ser. fiz., **38**, 2612 (1974); H.M. Sommermann, Ann. Phys. **151**, 163 (1983); D. Vautherin, N. Vinh Mau, Nucl.Phys. A **422**, 140 (1984); O. Civitarese, et al., Ann.Phys. **156**, 142 (1984); H. Sagawa, G.F. Bertsch, Phys. Lett. B **146**, 138 (1984)
9. P. Ring, J. Egido, J.Phys.G: Part. and Nucl., **19**, 1 (1993)
10. C. Yannouleas, S. Yang, Nucl. Phys. A **455**, 40 (1986); P.-F. Bortignon, et al., Nucl.Phys. A **460**, 149 (1986); E.C. Seva and H.M. Sofia, Phys. Rev. C **56**, 3107 (1997); N. Giovanardi, P.F. Bortignon, R.A. Broglia, Nucl.Phys. A **641**, 95 (1998); D. Lacroix, Ph. Chomaz, S. Ayik, Phys.Rev. C **58**, 2154 (1998)
11. D.S. Kosov, A.I. Vdovin, J. Wambach, in *Proceedings of the International Conference "Nuclear Structure and Related Topics" (Dubna, 1997*, edited by S.N. Ershov, R.V. Jolos, V.V. Voronov, (E4-97-327, Dubna, 1997) p.254
12. Ken -ji Hara, Prog. Theor. Phys. **32**, 88 (1964)
13. D.J. Rowe, Phys. Rev. **175**, 1283 (1968)
14. K. Ikeda, T. Udagawa and H. Yamamura, Prog. Theor. Phys. **33**, 22 (1965); J. da Providencia, Nucl. Phys. A **108**, 589 (1968); R.V. Jolos and W. Rybarska, Z. Physik A **296**, 73 (1980); W. Navrotska-Rybarska, O. Stoynova, Ch. Stoyanov, Yad. Fiz. **33**, 1494 (1980); C.H. Johnson, C. Mahaux, Phys. Rev. C **38**, 2589 (1988); A. Klein, N.R. Walet and C. Do Dang, Nucl. Phys. A **535**, 1 (1991); F. Catara, G. Piccitto, M. Sambataro, Phys. Rev. B **54**, 17536 (1996)
15. D. Karadjov, V. Voronov, F. Catara, Phys. Lett. B **306**, 197 (1993); J. Toivanen, J. Suhonen, Phys. Rev. Lett. **75**, 410 (1995)
16. A.I. Vdovin, D.S. Kosov, W. Nawrocka, Theor. Math. Phys. **111**, 613 (1997)
17. H.J. Lipkin, N. Meshkov, A.J. Glick, Nucl.Phys. **62**, 188 (1965)
18. A.N. Storozhenko, D.S. Kosov, A.I. Vdovin, Yad. Fiz. **62**, 63 (1999)
19. S.V. Tyablikov, *Methods in the Quantum Theory of Magnetism* (Plenum Press, New York 1967)
20. P. Schuck, S. Ethofer, Nucl. Phys. A **212**, 269 (1973); J. Dukelsky, P. Schuck, Nucl. Phys. A **512**, 466 (1990); J. Dukelsky, P. Schuck, Phys. Lett. B **387**, 233 (1996); J. Dukelsky, G. Röpke, P. Schuck, Nucl. Phys. A **628**, 17 (1998)
21. N.R. Walet, A. Klein, Nucl. Phys. A **510**, 261 (1990)
22. A. Kuriyama, et al., Prog. Theor. Phys. **87**, 911 (1992)
23. A. Kuriyama, et al., Prog. Theor. Phys. **94**, 1039 (1995)
24. A. Kuriyama, et al., Prog. Theor. Phys. **95**, 339 (1996)
25. A. Kuriyama, et al., Prog. Theor. Phys. **96**, 125 (1996)
26. G. Puddu, P.-F. Bortignon, R.A. Broglia, Ann. Phys. **206**, 409 (1991); Y. Alhassid, B.W. Bush, Nucl. Phys. A **549**, 43 (1992); S.Y. Tsay Tzeng, et al., Nucl. Phys. A **590**, 277 (1994); N. Canosa, R. Rossignoli, Phys. Rev. C **56**, 791 (1997)
27. P. Ring, P. Schuck, *The Nuclear Many-Body Problem*, (Springer-Verlag, New York 1980)
28. Steven E. Koonin, *Computational Physics*, (Addison-Wesley Publishing Company 1986)